

PARTICLE PATHS IN SMALL AMPLITUDE SOLITARY WAVES WITH NEGATIVE VORTICITY

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ABSTRACT. We investigate the particle trajectories in solitary waves with vorticity, where the vorticity is assumed to be negative and decrease with depth. We show that the individual particle moves in a similar way as that in the irrotational case if the underlying laminar flow is favorable, that is, the flow is moving in the same direction as the wave propagation throughout the fluid, and show that if the underlying current is not favorable, some particles in a sufficiently small solitary wave move to the opposite direction of wave propagation along a path with a single loop or hump.

1. INTRODUCTION

The *water-wave problem* concerns the gravity-driven flow of a perfect fluid of unit density; the flow is bounded below by a rigid horizontal bottom $\{Y = -d\}$ and above by a free surface $\{Y = \eta(X, t)\}$, where η depends upon the horizontal spatial coordinate X and time t . *Steady waves* are waves which propagate from left to right with constant speed c and without change of shape, so that $\eta(X, t) = \eta(X - ct)$. *Solitary waves* are steady waves which have the property that $\eta(X - ct) \rightarrow 0$ as $X - ct \rightarrow \pm\infty$. We consider in this paper the particle trajectories in a fluid as a solitary wave propagates on the free surface, assuming that the flow admits a negative vorticity decreasing with depth.

There have been a series of works concentrating on the study of solitary waves, in the setting of both irrotational flows [1, 2, 3, 6, 7] etc., and rotational flows which become active only in the last few years. One of the interests in the above works is the description of individual particle path in the fluid. In *irrotational* flow, particle paths underneath a solitary wave are investigated in [4, 6], both for the smooth solitary wave and the solitary wave of greatest height for which the crest is a stagnation point. It was shown in [6] that in an irrotational solitary water wave, each particle is transported in the wave direction but slower than the wave speed; as the solitary wave propagates, all particles located ahead of the wave crest are lifted while those behind have a downward motion, with the particle trajectory having asymptotically the same height above the flat bed. For *rotational* flow, some results on particle paths under *periodic* waves are obtained (cf. [11, 10, 16, 18] for instance), following a series of works on the corresponding results in irrotational case (see [5, 9, 13] etc.).

Recently, rigorous existence results on small-amplitude solitary waves with arbitrary vorticity distribution were obtained in [14] and in [12], using generalized implicit function theorem of Nash-Moser type and spatial dynamics method, respectively. The solitary waves established in [14, 12] are of elevation and decays exponentially to a horizontal laminar flow far up- and downstream. The study of solitary waves of large-amplitude with an arbitrary distribution of vorticity remains open. So this arises the question of particle paths in a rotational small-amplitude solitary wave. Following the pattern in [6] for irrotational case, we prove in this work the corresponding results on particle paths in rotational solitary waves

by using the properties of solutions obtained in [14]. We consider in this work only solitary waves with negative vorticity, however with modifications our arguments can be applied also for positive vorticity.

Precisely we show that if the vorticity is negative and increasing from bottom to surface of the flow, and if the underlying current is favorable, i.e., is moving throughout the fluid in the same direction as the wave, then as time goes on the particle moves similarly as in the irrotational case [6]. We also show that if the underlying current is not favorable, then in solitary waves with sufficiently small amplitude, some particles above the flat bed move to the opposite direction of wave propagation along a path with a single loop or a single hump. Note that this single-loop kind of path does not exist in the irrotational case. In [14], as in most of the works on waves with vorticity, the author considered only waves that are not near breaking or stagnation, i.e., the speed of an individual fluid particle is far less than that of the wave itself throughout the fluid domain. We consider only such waves as well. We do not consider the case of wave with stagnation which is however studied in [6] in the irrotational case.

The paper is organized as follows. In Section 2 we present the mathematical formulation for the solitary waves and recall from [14] some useful properties of the established solitary wave solutions. Section 3 contains some conclusions on the vertical and horizontal velocity that are relevant for our purposes. The main result is presented and proved in Section 4. In the final section we give two examples which lie in our settings.

2. PRELIMINARIES

We first describe the governing equations for rotational solitary water waves and then recall some properties available on their solitary wave solutions.

2.1. The governing equations for rotational solitary water waves. Choose Cartesian coordinates (X, Y) so that the horizontal X -axis is in the direction of wave propagation and the Y -axis points upwards. Consider steady waves traveling at constant speed $c > 0$. In the frame of reference moving with the wave, which is equivalent to the change of variables $(X - ct, Y) \mapsto (x, y)$, we use

$$\Omega_\eta = \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\}, \quad 0 < d < \infty,$$

to denote the stationary fluid domain and $(u(x, y), v(x, y))$ to denote the velocity field, and define the *stream function* $\psi(x, y)$ by $\psi(x, \eta(x)) = 0$ and

$$\psi_y = u - c, \quad \psi_x = -v. \tag{1}$$

Consider also only waves that are not near breaking or stagnation, so that $\psi_y(x, y) \leq -\delta < 0$ in $\bar{\Omega}_\eta$ for some $\delta > 0$, which implies that the vorticity $\omega = v_x - u_y$ is globally a function of the stream function ψ , denoted by $\gamma(\psi)$; see [8]. The solitary-wave problem is then, for given $p_0 < 0$ and $\gamma \in C^1([0, -p_0]; \mathbb{R})$, to find a real parameter λ , a domain Ω_η and a function

$\psi \in C^2(\bar{\Omega}_\eta)$ such that

$$\psi_y < 0, \quad (x, y) \in \bar{\Omega}_\eta, \quad (2)$$

$$\Delta\psi = -\gamma(\psi), \quad (x, y) \in \Omega_\eta, \quad (3)$$

$$|\nabla\psi|^2 + 2gy = \lambda, \quad y = \eta(x), \quad (4)$$

$$\psi = 0, \quad y = \eta(x), \quad (5)$$

$$\psi = -p_0, \quad y = -d, \quad (6)$$

$$\eta(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (7)$$

$$\psi_x(x, y) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \text{ uniformly for } y. \quad (8)$$

Here $g > 0$ is the gravitational constant of acceleration,

$$p_0 = \int_{-d}^{\eta(x)} \psi_y(x, y) dy < 0$$

is the relative mass flux (independent of x), and the boundary conditions (7) and (8) express that the wave profile approaches a constant level of depth and the flow is almost horizontal in the far field, respectively. Moreover we require that the nontrivial solitary wave is of positive elevation, i.e., $\eta(x) > 0$ for all $x \in \mathbb{R}$. It is therefore symmetric about its single crest and admits a strictly monotone wave profile on either side of this crest (see [15]). Assuming the wave crest is located at $x = 0$, the solitary-wave problem (2)-(8) is thus supplemented with the symmetry and monotonicity conditions

$$\psi(-x, y) = \psi(x, y), \quad \eta(-x) = \eta(x), \quad \text{and } \eta'(x) < 0 \text{ for } x > 0. \quad (9)$$

We refer to [14, 8] for more details on the derivation of the system (2)-(8).

2.2. Rotational solitary water waves. We collect some properties of the solitary wave solutions established in [14]. Let

$$\Gamma(p) = \int_0^p \gamma(-s) ds \quad \text{and} \quad \Gamma_{\min} = \min_{p \in [p_0, 0]} \Gamma(p) \leq 0. \quad (10)$$

Given $p_0 < 0$ and $\gamma \in C^1([0, -p_0]; \mathbb{R})$, for each $\lambda \in (-2\Gamma_{\min}, \infty)$ the system (2)-(8) admits a trivial solution pair $(\eta(x), \Psi(y))$ defined on $\bar{\Omega}_0$, where $\eta(x) \equiv 0$, the stream function $\Psi(y)$ is x -independent and is the inverse of the function

$$y(\Psi) = \int_{p_0}^{-\Psi} \frac{dp}{\sqrt{\lambda + 2\Gamma(p)}} - d,$$

and the fluid domain

$$\Omega_0 = \{(x, y) \in \mathbb{R}^2 : -d < y < 0\}, \quad d = \int_{p_0}^0 \frac{dp}{\sqrt{\lambda + 2\Gamma(p)}}.$$

The corresponding relative horizontal velocity and vertical velocity are thus given by

$$U(y) - c = \Psi_y(y) = -\sqrt{\lambda + 2\Gamma(-\Psi(y))}, \quad V(x, y) = -\Psi_x(y) \equiv 0. \quad (11)$$

Note that $\Psi(0) = 0$ and $\Psi(-d) = -p_0$. Thus

$$U(0) = c - \sqrt{\lambda} \quad \text{and} \quad U(-d) = c - \sqrt{\lambda + 2\Gamma(p_0)}.$$

Throughout this paper, we adopt the terminology in [9] to say this underlying trivial flow is *favorable* if $U(y) \geq 0$ for all $y \in [-d, 0]$, is *adverse* if $U(y) < 0$ for all $y \in [-d, 0]$, and is *mixed* if $U(y)$ changes sign. Note that favorable flow moves in the same direction as the

wave propagation (i.e., to the right), while adverse flow moves in the opposite direction of the wave propagation.

To ensure the existence of nontrivial small amplitude solitary waves, the parameter λ must be chosen to satisfy $\lambda > \lambda_c$ but close to λ_c , where $\lambda_c > -2\Gamma_{\min}$ is the unique solution of

$$\int_{p_0}^0 \frac{dp}{(\lambda_c + 2\Gamma(p))^{3/2}} = \frac{1}{g}. \quad (12)$$

For each such a given λ and for given $p_0 < 0$, it was shown in [14] that there exists a nontrivial small amplitude solitary-wave solution pair $(\eta(x), \psi(x, y))$ to (2)-(8) defined on $\bar{\Omega}_\eta$, with $\eta(x)$ satisfying

$$|\eta(x)| + |\eta'(x)| + |\eta''(x)| \leq (\lambda - \lambda_c)r \quad \text{for all } x \in \mathbb{R} \quad (13)$$

and for some constant $r > 0$ independent of λ , and the corresponding horizontal velocity satisfying the following properties:

$$\begin{aligned} (\mathcal{P1}, \eta(x)) &\rightarrow c - \sqrt{\lambda} = U(0), \quad \text{as } |x| \rightarrow \infty; \\ (\mathcal{P2}, y) &\rightarrow U(y) \quad \text{as } |x| \rightarrow \infty \text{ uniformly for } y. \end{aligned}$$

The first property is obvious since by (4), (7) and (8), we have $(u(x, \eta(x)) - c)^2 = \psi_y^2(x, \eta(x)) \rightarrow \lambda$ as $|x| \rightarrow \infty$, which equivalently gives (P1) as we have assumed $u(x, y) - c = \psi_y(x, y) < 0$ in $\bar{\Omega}_\eta$, while the property (P2) can be deduced from the construction of solitary wave solutions; see [14]. Indeed for $\lambda = \lambda_c + \varepsilon$ with $\varepsilon > 0$, there exists a function $w^\varepsilon(q, p)$, whose derivatives with respect to (q, p) up to order 2 tend to 0 uniformly for p as $|q| \rightarrow \infty$, such that the horizontal velocity is determined by

$$u(x, y) = c - \frac{1}{(\lambda + 2\Gamma(-\psi(x, y)))^{-1/2} + \varepsilon w_p^\varepsilon(\sqrt{\varepsilon}x, -\psi(x, y))},$$

where $w_p^\varepsilon(q, p)$ denotes differentiation in the p -variable.

We conclude this section by recalling some properties of streamlines. Due to $\psi_y < 0$ throughout $\bar{\Omega}_\eta$, we have that for all $p \in [-p_0, 0]$ the streamline

$$\{(x, y) : \psi(x, y) = -p\}$$

is a smooth curve $y = \sigma_p(x)$. Note that

$$\sigma_0(x) = \eta(x), \quad \sigma_{p_0}(x) = -d \quad \text{and} \quad \sigma_p'(x) = -\frac{\psi_x(x, \sigma_p(x))}{\psi_y(x, \sigma_p(x))}. \quad (14)$$

Observing the fact that $\frac{1}{|\psi_y|} = \frac{1}{c-u}$ is bounded and that $\psi_x(x, y) \rightarrow 0$ uniformly for y as $|x| \rightarrow \infty$, we deduce that for each fixed $y_0 \in [0, \eta(0)]$ the streamline $y = \sigma_p(x)$ with $p = -\psi(0, y_0)$, passing through the point $(0, y_0)$, has an asymptote $y = l(y_0)$ as $|x| \rightarrow \infty$, with $l(\eta(0)) = 0$ and $l(-d) = -d$.

3. VERTICAL AND HORIZONTAL VELOCITY

We first divide the fluid domain Ω_η into two components

$$\Omega_- = \{(x, y) \in \mathbb{R}^2 : x < 0, 0 < y < \eta(x)\} \quad \text{and} \quad \Omega_+ = \{(x, y) \in \mathbb{R}^2 : x > 0, 0 < y < \eta(x)\},$$

and denote their boundaries by

$$S_- = \{(x, y) \in \mathbb{R}^2 : x < 0, y = \eta(x)\}, \quad B_- = \{(x, y) \in \mathbb{R}^2 : x < 0, y = -d\},$$

respectively

$$S_+ = \{(x, y) \in \mathbb{R}^2 : x > 0, y = \eta(x)\}, \quad B_+ = \{(x, y) \in \mathbb{R}^2 : x > 0, y = -d\}.$$

Lemma 3.1. *Suppose that $\gamma'(p) \leq 0$ for all $p \in [0, |p_0|]$. Then*

- (a) $v(x, -d) = 0$ for all $x \in \mathbb{R}$, and $v(0, y) = 0$ for $y \in [-d, \eta(0)]$;
- (b) $v(x, y) < 0$ if $(x, y) \in \Omega_- \cup S_-$, and $v(x, y) > 0$ if $(x, y) \in \Omega_+ \cup S_+$;
- (c) $v_y(x, -d) < 0$ if $x < 0$, and $v_y(x, -d) > 0$ if $x > 0$;
- (d) $v_x(0, y) > 0$ for $y \in (-d, \eta(0))$.

Proof. Since $v = -\psi_x$, the first result follows from (6) and the symmetry property $\psi(-x, y) = \psi(x, y)$ in (9).

Next we only prove the lemma for $x > 0$ and the results for $x < 0$ can be proved similarly. Differentiating (5) with respect to x gives $v = -\psi_x = \psi_y \eta'(x)$, from which follows $v > 0$ for $(x, y) \in S_+$ as $\psi_y < 0$ and $\eta'(x) < 0$ for $x > 0$ in view of (2) and (9). To prove $v > 0$ in Ω_+ , we assume first on the contrary that there exists a point $(x_0, y_0) \in \Omega_+$ such that $v(x_0, y_0) = -\varepsilon < 0$. Then we can find a bounded domain

$$\Omega_{+,k} = \{(x, y) \in \mathbb{R} : 0 < x < k, -d < y < \eta(x)\}, \quad k \in \mathbb{R}^+,$$

such that $(x_0, y_0) \in \Omega_{+,k}$. Moreover in view of (8) we can choose k sufficiently large so that $v(k, y) > -\varepsilon$ for $y \in [-d, \eta(k)]$. This means that v attains its minimum at the interior point (x_0, y_0) of $\Omega_{+,k}$, which contradicts to the strong maximum principle applied to v on the domain $\bar{\Omega}_{+,k}$, as v satisfies $\Delta v + \gamma'(\psi)v = 0$ with $\gamma'(p) \leq 0$ by differentiating (3). Therefore we have $v \geq 0$ in Ω_+ . If $v = 0$ at a point (x_0, y_0) of Ω_+ . Again we can choose a bounded domain $\Omega_{+, \hat{k}}$ containing (x_0, y_0) . Then $v \geq 0$ on the boundary of $\Omega_{+, \hat{k}}$. We can thus apply the strong maximum principle on $\Omega_{+, \hat{k}}$ once more to conclude that $v \equiv 0$ on $\bar{\Omega}_{+, \hat{k}}$, which contradicts $v > 0$ on the half surface S_+ . This proves $v > 0$ in Ω_+ .

Since v attains its minimum in $\bar{\Omega}_+$ on the half bed B_+ and on the crest line $x = 0$, Hopf's maximum principle ensures that $v_y > 0$ on B_+ and $v_x > 0$ on $\{(0, y); -d < y < \eta(0)\}$, completing the proof. \square

The above lemma combined with (14) and the fact that $\psi_y < 0$ gives

Corollary 3.2. *The streamline $y = \sigma_p(x)$ with $p \in (p_0, 0]$ satisfies that $\sigma'_p(x) > 0$ for $x < 0$ and $\sigma'_p(x) < 0$ for $x > 0$.*

Lemma 3.3. *Suppose that $\gamma(p), \gamma'(p) \leq 0$ for all $p \in [0, |p_0|]$, that $0 < \lambda - \lambda_c < \varepsilon$ with ε small such that nontrivial solitary wave exists, and that $c \geq \sqrt{\lambda + 2\Gamma(p_0)}$. Then $u(x, y) > 0$ for $x \in \bar{\Omega}_\eta$.*

Proof. We assume throughout this proof that $\gamma(p) \not\equiv 0$ for $p \in [0, |p_0|]$, since we already know that for $\gamma(p) \equiv 0$, i.e., the irrotational case, $u(x, y) > 0$ in $\bar{\Omega}_\eta$; see [6]. Recall that $U(y)$ is the horizontal velocity of the trivial laminar flow. Thus $U'(y) = \Psi_{yy}(x, y) = -\gamma(\Psi(y)) \geq 0$, and there exists an interval I such that the strict inequality holds for $y \in I$ since $\gamma(p) \not\equiv 0$ by assumption. This combined with (11) and the assumption $c \geq \sqrt{\lambda + 2\Gamma(p_0)}$ gives

$$U(y) \geq U(-d) = c - \sqrt{\lambda + 2\Gamma(p_0)} \geq 0 \text{ for } y \in [-d, 0], \text{ and } U(0) > 0.$$

It was proved in [17] that if $\gamma(p) \leq 0$ for all $p \in [0, |p_0|]$, then

$$\frac{d}{dx}u(x, \eta(x)) \geq 0 \text{ for } x < 0, \text{ and } \frac{d}{dx}u(x, \eta(x)) \leq 0 \text{ for } x > 0. \quad (15)$$

In other words, along the free surface u increases from $x = -\infty$ to the crest $x = 0$, and thereafter it is decreasing. On the bottom $B = \{(x, y) \in \mathbb{R}^2 : y = -d\}$, we have, in view of $u_x = -v_y = \psi_{xy}$ and Lemma 3.1-(c),

$$u_x(x, -d) > 0 \text{ for } x < 0, \text{ and } u_x(x, -d) < 0 \text{ for } x > 0. \quad (16)$$

Then the fact that $U(-d) \geq 0$ and $U(0) > 0$ together with the monotonicity properties (15) and (16) yields $u > 0$ on the free surface and on the bottom. Differentiating (3) with respect to y yields that u satisfies

$$\Delta u + \gamma'(\psi)u = \gamma'(\psi)c \leq 0.$$

Finally considering $U(y) \geq 0$ for $y \in [-d, 0]$ and the properties (P1)-(P2), we can deduce $u > 0$ in $\bar{\Omega}_\eta$ by using the strong maximum principle as in the proof of Lemma 3.1-(b), completing the proof. \square

The above lemma considers the case $c \geq \sqrt{\lambda + 2\Gamma(p_0)}$, that is the underlying flow is favorable. For the remaining cases, we can only derive some conclusions on the horizontal velocity u for some special vorticity function γ and for λ sufficiently close to the corresponding λ_c defined in (12). To be exact, we prove

Lemma 3.4. *Let λ_c be determined in (12) by the vorticity function $\gamma(p)$ with $p \in [0, |p_0|]$.*

(a) *If $\gamma'(p), \gamma''(p) \leq 0$ for $p \in [0, |p_0|]$ and $\gamma(0)\sqrt{\lambda_c} > -g$, then there exists $\varepsilon_1 > 0$ such that for $\lambda \in (\lambda_c, \lambda_c + \varepsilon_1)$, we have*

$$u_x > 0 \text{ for } (x, y) \in \Omega_-, \text{ and } u_x < 0 \text{ for } (x, y) \in \Omega_+;$$

(b) *If $\gamma(p) < 0$ for all $p \in [0, |p_0|]$, then there exists $\varepsilon_2 > 0$ such that for $\lambda \in (\lambda_c, \lambda_c + \varepsilon_2)$, we have*

$$u_y > 0 \text{ for } (x, y) \in \Omega.$$

Proof. (a) Similar results have been obtained for periodic waves in [10], so we will adapt the proof there to our present solitary wave case. We prove only the result for $(x, y) \in \Omega_+$ and the rest can be proved similarly. Note it suffices to prove that the stream function ψ satisfies

$$\psi_{xy} < 0 \text{ for } (x, y) \in \Omega_+.$$

Since $\psi(x, \eta(x)) = 0$, we have $\psi_x = -\eta'\psi_y$ on the surface, which combined with (4) gives

$$\psi_y(x, \eta(x)) = -\sqrt{(\lambda - 2g\eta)/(1 + \eta'^2)}.$$

Differentiating the above equality and $\psi_x = -\eta'\psi_y$ along the surface gives respectively

$$\psi_{xy} + \eta'\psi_{yy} = -\partial_x \sqrt{\frac{\lambda - 2g\eta}{1 + \eta'^2}} \quad \text{and} \quad \psi_{xx} + \eta'\psi_{yy} + 2\eta'\psi_{xy} + \eta''\psi_y = 0, \text{ on } y = \eta(x).$$

Moreover on the surface one has

$$\psi_{xx} + \psi_{yy} = -\gamma(0).$$

Combination of the above three equalities gives

$$\psi_{xy}(x, \eta(x)) = \frac{\eta' (g(1 - \eta'^4) + 2\eta''(\lambda - 2g\eta) + \gamma(0)\sqrt{\lambda - 2g\eta}(1 + \eta'^2)^{3/2})}{\sqrt{\lambda - 2g\eta}(1 + \eta'^2)^{5/2}}.$$

In view of (13) and $\gamma(0)\sqrt{\lambda_c} > -g$, we have when λ is sufficiently close to λ_c that

$$g(1 - \eta'^4) + 2\eta''(\lambda - 2g\eta) + \gamma(0)\sqrt{\lambda - 2g\eta}(1 + \eta'^2)^{3/2} > 0.$$

This gives $\psi_{xy} < 0$ on the surface since $\eta' < 0$ for $x > 0$ and $\sqrt{\lambda - 2g\eta}(1 + \eta'^2)^{5/2} > 0$. Since $v(0, y) = 0$, we have $\psi_{xy} = -v_y = 0$ on the line $x = 0$. On the bottom $\psi_{xy} < 0$ holds due to Lemma 3.1-(c). And $\psi_{xy} = u_x \rightarrow 0$ as $x \rightarrow \infty$ since $u(x, y) \rightarrow U(y)$ as $x \rightarrow \infty$. Finally ψ_{xy} satisfies

$$(\Delta + \gamma')\psi_{xy} = -\gamma''\psi_x\psi_y \geq 0.$$

The conclusion follows from the maximum principle.

(b) Recall from [14] that for $\lambda = \lambda_c + \varepsilon$, there exists a function $w^\varepsilon(q, p)$ such that $u(x, y)$ is determined by

$$u(x, y) = c - \frac{1}{(\lambda + 2\Gamma(-\psi(x, y)))^{-1/2} + \varepsilon w_p^\varepsilon(\sqrt{\varepsilon}x, -\psi(x, y))}, \quad (17)$$

where $w_p^\varepsilon(q, p)$ stands for the differentiation of the function $w^\varepsilon(q, p)$ with respect to p , and the family $\{w^\varepsilon(q, p); 0 \leq \varepsilon < 1\}$ satisfies that for ε small

$$\sup \left\{ \left| \partial_q^j \partial_p^k w^\varepsilon(q, p) \right|; j + k \leq 2, (q, p) \in \mathbb{R} \times [p_0, 0] \right\} \leq r, \quad (18)$$

with $r > 0$ some constant independent of ε . Differentiating (17) with respect to y gives

$$u_y = \frac{-\psi_y \left(-\gamma(\psi) (\lambda + 2\Gamma(-\psi))^{-3/2} + \varepsilon w_{pp}^\varepsilon(\sqrt{\varepsilon}x, -\psi) \right)}{\left((\lambda + 2\Gamma(-\psi))^{-1/2} + \varepsilon w_p^\varepsilon(\sqrt{\varepsilon}x, -\psi) \right)^2}.$$

Since $\gamma(p) < 0$, if we denote $\gamma_{\max} = \max_{p \in [0, |p_0|]} \gamma(p)$, then $\gamma_{\max} < 0$ and $\Gamma_{\max} = \max_{p \in [p_0, 0]} \Gamma(p) > 0$. As a result, observing

$$-\gamma(\psi) (\lambda + 2\Gamma(-\psi))^{-3/2} + \varepsilon w_{pp}^\varepsilon(\sqrt{\varepsilon}x, -\psi) \geq -\gamma_{\max} (\lambda + 2\Gamma_{\max})^{-3/2} + \varepsilon w_{pp}^\varepsilon(\sqrt{\varepsilon}x, -\psi)$$

and

$$\lim_{\varepsilon \rightarrow 0} \left(-\gamma_{\max} (\lambda + 2\Gamma_{\max})^{-3/2} + \varepsilon w_{pp}^\varepsilon(\sqrt{\varepsilon}x, -\psi) \right) = -\gamma_{\max} (\lambda_c + 2\Gamma_{\max})^{-3/2}$$

due to (18), we get, in view of $-\gamma_{\max} (\lambda_c + 2\Gamma_{\max})^{-3/2} > 0$, that

$$-\gamma(\psi) (\lambda + 2\Gamma(-\psi))^{-3/2} + \varepsilon w_{pp}^\varepsilon(\sqrt{\varepsilon}x, -\psi) > 0$$

when ε is sufficiently small. This combined with $-\psi_y > 0$ gives $u_y > 0$ for ε sufficiently small. \square

Lemma 3.5. *If $\gamma(p) < 0, \gamma'(p), \gamma''(p) \leq 0$ for $p \in [0, |p_0|]$ and $\gamma(0)\sqrt{\lambda_c} > -g$, then for $\lambda \in (\lambda_c, \lambda_c + \varepsilon_0)$ with $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, along every streamline $y = \sigma_p(x)$ with $p \in (p_0, 0)$, the horizontal velocity u is strictly decreasing in Ω_+ and strictly increasing in Ω_- as a function of x .*

Proof. Since

$$\frac{d}{dx}u(x, \sigma_p(x)) = u_x + u_y \sigma'_p(x) = u_x + u_y \frac{v}{u - c}$$

due to (14), the conclusion follows immediately from Lemma 3.1 and Lemma 3.4. \square

4. MAIN RESULT

Recalling that $y = l(y_0)$ is the streamline asymptote introduced at the end of Section 2, we are now ready to state our main result describing the particle trajectories in a solitary wave with negative vorticity.

Theorem 4.1. *Let the flux $p_0 < 0$, the vorticity function $\gamma \in C^2(0, |p_0|)$ be given, and let λ_c be determined as in (12). Assume that $\gamma(p) < 0, \gamma'(p) \leq 0$ for all $p \in [0, |p_0|]$, and that $\lambda > \lambda_c$ such that nontrivial solitary wave solutions to (2)-(8) exist. Then the following results hold.*

- (a) *Any particle above the bed reaches at some instant t_0 the location (X_0, Y_0) below the wave crest $(X_0, \eta(0))$;*
- (b) *For $c \geq \sqrt{\lambda + 2\Gamma(p_0)}$, as time t runs on $(-\infty, t_0)$, the particle above the flat bed moves to the right and upwards, while for $t > t_0$ the particle moves to the right and downwards, as in Figure 1-(a); the particle on the flat bed moves in a straight line to the right at a positive speed;*
- (c) *If assume additionally that $\gamma(0)\sqrt{\lambda_c} > -g$ and $\gamma''(p) \leq 0$ for all $p \in [0, |p_0|]$, then there exists $\varepsilon_0 > 0$ such that for $\lambda \in (\lambda_c, \lambda_c + \varepsilon_0)$ and*
 - (i) *for $\sqrt{\lambda} < c < \sqrt{\lambda + 2\Gamma(p_0)}$, there does not exist a single pattern for all the particles above the flat bed; if $u(x, -d) \geq 0$, depending on the relation between the asymptote $y = l(Y_0)$ and the zero point y_* of $U(y)$, some particles move to the right along a path with a single hump as described in (b), some move along a single-loop path to the left, as in Figure 1-(b); if $u(x, -d) < 0$, there are additionally some particles moving to the left along a single-hump path; see Figure 1-(c);*
 - (ii) *for $c \leq \sqrt{\lambda}$, depending on the signs of $u(0, \eta(0))$ and $u(0, -d)$, there are three possibilities for the particles above the flat bed; see Figure 3;*
 - (iii) *for a particle on the flat bed in both the cases (i) and (ii), if $u(x, -d) \leq 0$ it moves to the left in a straight line, while if $u(x, -d) > 0$ it firstly has a backward-forward pattern of motion and then moves to the left in a straight line;*
- (d) *The particle trajectory is strictly above the asymptote $Y = l(Y_0)$ of the streamline $Y = \sigma_p(X - ct)$ with $p = -\psi(0, Y_0)$.*

Proof. The path (past and future) $(X(t), Y(t))$ of a particle with location $(X(0), Y(0))$ at time $t = 0$ is given by the solution of the non-autonomous system

$$\begin{cases} \dot{X} = u(X - ct, Y), \\ \dot{Y} = v(X - ct, Y). \end{cases}$$

Working in the moving frame $x = X - ct$ and $y = Y$, we transform the above system into

$$\begin{cases} \dot{x} = u(x, y) - c, \\ \dot{y} = v(x, y). \end{cases} \quad (19)$$

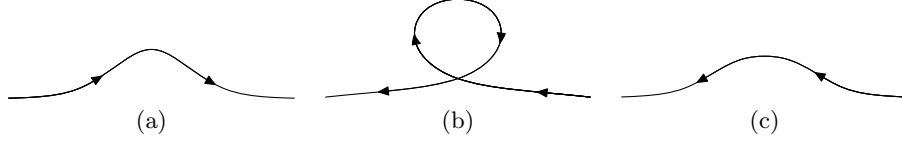


FIGURE 1. Particle path with a single: (a) hump to the right; (b) loop to the left; (c) hump to the left.

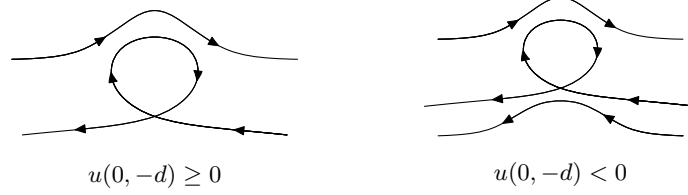


FIGURE 2. Particle path above the flat bed in a small solitary wave with a mixed underlying current.

This is a Hamiltonian system with Hamiltonian $\psi(x, y)$ in view of (1), meaning that the solutions $(x(t), y(t))$ of (19) lie on the streamlines. All solutions of (19) are defined globally in time since the boundedness of the right-hand side prevents blow-up in finite time.

(a) Since in the moving frame the wave crest is assumed to be located at $x = 0$ in our setting, the sign of $x(t)$ describes the position of the particle with respect to the wave crest at time t : the particle is exactly below the crest if $x(t) = 0$, is ahead of the crest if $x(t) > 0$ while is behind the crest if $x(t) < 0$. Note that $\dot{x} = u(x, y) - c \leq -\delta < 0$ throughout $\bar{\Omega}_\eta$. This uniform upper bound on \dot{x} implies that $x(t) \rightarrow \mp\infty$ as $t \rightarrow \pm\infty$ and there is a unique time t_0 such that $x(t_0) = 0$. That is, to each fluid particle moving within the water there corresponds a unique time $t_0 \in \mathbb{R}$ so that at $t = t_0$ the particle is exactly below the wave crest, while afterwards it is located behind the wave crest, the wave crest being behind the particle for $t < t_0$.

In the subsequent of the proof, we always assume that the particle is located below the crest at time $t = t_0$ at the location $(X_0, Y_0) = (X(t_0), Y(t_0))$, or equivalently $(x(t_0), y(t_0)) = (0, Y_0)$ which implies in fact $X_0 = ct_0$.

(b) Assume $c \geq \sqrt{\lambda + 2\Gamma(p_0)}$. For a particle located above the flat bed, we prove only the statement for $t > t_0$ and the case $t < t_0$ can be proved similarly. Since $x(t)$ is strictly decreasing, one has $X(t) - ct = x(t) < 0$ for $t > t_0$, which combined with Lemma 3.1-(b) implies $\dot{Y} < 0$ for $t > t_0$. Furthermore we have $\dot{X} > 0$ for all time by Lemma 3.3. That means the particle moves to the right and downwards as time runs on $(t_0, +\infty)$. The statement for particles on the bed follows from Lemma 3.1-(a) and Lemma 3.3. The particle path for this case is depicted in Figure 1-(a).

(c)-(i) Since $\sqrt{\lambda} < c < \sqrt{\lambda + 2\Gamma(p_0)}$, we have $U(0) > 0$ and $U(-d) < 0$, thus $U(y)$ has a unique zero point, say at y_* . Assume first that $u(0, -d) \geq 0$. The monotonicity properties of u provided by Lemma 3.4-(a) and (16) show that in $\bar{\Omega}_+$ the level set $\{u = 0\}$ consists of a continuous curve \mathcal{C}_+ in the moving frame. The curve is confined between $y = -d$ and $y = y_*$, and can be parameterized by $x = h(y)$ with $h'(y) > 0$, $h(y) \rightarrow +\infty$ as $y \rightarrow y_*$, and

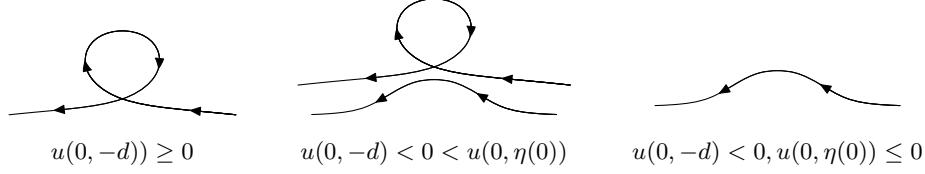


FIGURE 3. Particle path above the flat bed in a small solitary wave with an adverse underlying current.

$h(-d) \geq 0$ with the equality holding when $u(x, -d) = 0$. In $\bar{\Omega}_-$ the level set $\{u = 0\}$ is given by the reflection \mathcal{C}_- of the curve \mathcal{C}_+ across the line $x = 0$. Below the curve \mathcal{C}_+ and \mathcal{C}_- we have $u < 0$ (including the bottom), while above and between the two curves we have $u > 0$. Furthermore, in view of Lemma 3.5, if a streamline and the curve \mathcal{C}_+ (rep. \mathcal{C}_-) intersect, they intersect exactly once.

Recall that $(x(t), y(t))$ lies on the streamline $y = \sigma_p(x)$ with $p = -\psi(0, Y_0)$ since $(x(t_0), y(t_0)) = (0, Y_0)$, and recall that $y = l(Y_0)$ is the asymptote of the streamline passing through the point $(0, Y_0)$. In virtue of Corollary 3.2, the path $(x(t), y(t))$ is located below $y = Y_0$ and above the asymptote $y = l(Y_0)$ for all time t . If $l(Y_0) \geq y_*$, then we have from the above arguments that $u > 0$ for all time t , while $v < 0$ if $t > t_0$ and $v > 0$ if $t < t_0$. This is the same situation as described in (b), so that the particle moves to the right along a path with a single hump, as depicted in Figure 1-(a).

If $l(Y_0) < y_*$, the particle trajectories above the flat bed are as shown in Figure 1-(b). Indeed, in this case, as time t increases from $-\infty$, the path $(x(t), y(t))$ intersects successively the curve \mathcal{C}_+ from below at $t = t_+$, the vertical line $x = 0$ from right at $t = t_0$, and the curve \mathcal{C}_- from above at $t = t_-$. In the time interval $t \in (-\infty, t_+) \cup (t_-, +\infty)$ we know that $u < 0$ so that in the physical variables (X, Y) the particle $(X(t), Y(t))$ moves to the left. In the time interval $t \in (t_+, t_-)$ we have $u > 0$ so that $(X(t), Y(t))$ moves to the right. Also we have $v > 0$ when $t < t_0$ so that $(X(t), Y(t))$ moves up, while when $t > t_0$ we have $v < 0$ so that $(X(t), Y(t))$ moves down. Thus in this case the particle above the flat bed moves to the left along a path with a single loop.

It remains to treat the case $u(0, -d) < 0$. Note $u(0, \eta(0)) > 0$ in virtue of $U(0) > 0$ and (15). Thus, in view of Lemma 3.4-(b), there exists a unique $\tilde{y} \in (-d, \eta(0))$ such that $u(0, \tilde{y}) = 0$. Observe $\tilde{y} < y_*$ since $u_x < 0$ in Ω_+ and $u(x, y) \rightarrow U(y)$ as $x \rightarrow \infty$. The corresponding curve \mathcal{C}_+ is now located between $y = y_*$ and $y = \tilde{y}$, intersecting the line $x = 0$ at $y = \tilde{y}$. For $Y_0 > \tilde{y}$, the paths $(X(t), Y(t))$ are similar as encountered in the case when $u(0, -d) \geq 0$. While for $Y_0 \leq \tilde{y}$, we have $(x(t), y(t))$ is below \mathcal{C}_+ and \mathcal{C}_- , so that $u < 0$ for all the time. As a result, the particles move to the left along a single-hump path; see Figure 1-(c).

We depict all the possible trajectories in this case in Figure 2.

(c)-(ii) This case can be treated similarly as that in the above, so we omit the details and show the particle trajectories in Figure 3.

(c)-(iii) The conclusions for particles on the flat bed follow from Lemma 3.1-(a) and (16).

(d) Observing $\dot{Y} = v(X - ct, Y)$, $X(t) - ct = x(t) \rightarrow -\infty$ as $t \rightarrow +\infty$ and $v(x, y)$ converges to 0 as $|x| \rightarrow \infty$ uniformly in y , we have the existence of some α such that $\lim_{t \rightarrow +\infty} Y(t) = \alpha$. Since $\psi(x, y)$ is the Hamiltonian function of the system (19), we have

$$\psi(x(t), y(t)) = \psi(x(t_0), y(t_0)) = \psi(0, Y_0).$$

Thus $y(t) = \sigma_p(x(t))$ with $p = -\psi(0, Y_0)$. Consequently

$$\lim_{t \rightarrow +\infty} Y(t) = \lim_{t \rightarrow +\infty} y(t) = \lim_{t \rightarrow +\infty} \sigma_p(x(t)) = l(Y_0).$$

The proof is completed. \square

Remark For the path with a single loop, if we define the size of the loop by the net horizontal distance moved by the particle between the two instants t_+ and t_- when its horizontal velocity changes sign, that is,

$$X(t_-) - X(t_+),$$

then the size decreases with depth. In fact it can be computed by

$$X(t_-) - X(t_+) = \int_{t_+}^{t_-} \frac{dX}{dt} dt = \int_{t_+}^{t_-} u(x, \sigma_p(x)) dt.$$

Since $u_y > 0$ and $\frac{d\sigma_p(x)}{dp} = -\frac{1}{\psi_y} > 0$, we have $X(t_-) - X(t_+)$ decreases as p decreases.

5. EXAMPLES

In this final section we give two examples of the vorticity function which satisfy the conditions imposed in our main theorem.

5.1. Negative constant vorticity. In the case of negative constant vorticity $\gamma(p) = -\omega_0$ for $p \in [0, |p_0|]$ with $\omega_0 > 0$, we have obviously $\gamma'(p), \gamma''(p) \leq 0$. So it remains to verify $\gamma(0)\sqrt{\lambda_c} > -g$, or equivalently, $\lambda_c < g^2/\omega_0^2$. Recall from (12) that $\lambda_c > -2\Gamma_{\min}$ is such that

$$\int_{p_0}^0 \frac{dp}{(\lambda_c + 2\Gamma(p))^{3/2}} = \frac{1}{g}$$

holds, where $\Gamma(p) = -\omega_0 p$ and $\Gamma_{\min} = 0$ in this case by (10). Set

$$F(\lambda) = \int_{p_0}^0 \frac{dp}{(\lambda - 2\omega_0 p)^{3/2}}.$$

Then direct computation shows that $F'(\lambda) < 0$, $F(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow 0+$, and

$$\lim_{\lambda \rightarrow g^2/\omega_0^2} F(\lambda) = \frac{1}{g} - \frac{1}{\sqrt{g^2 - 2\omega_0^3 p_0}} < \frac{1}{g}.$$

Thus there exists a unique $\lambda_c \in (0, g^2/\omega_0^2)$ such that $F(\lambda_c) = 1/g$.

5.2. Negative affine linear vorticity. For the vorticity $\gamma(p) = -ap + b$ with $a > 0$ and $b < 0$, we have obviously $\gamma(p) < 0$, $\gamma'(p), \gamma''(p) \leq 0$ for $p \in [0, |p_0|]$ and

$$F(\lambda) = \int_{p_0}^0 \frac{dp}{(\lambda + 2\Gamma(p))^{3/2}} = \int_{p_0}^0 \frac{dp}{(\lambda + ap^2 + 2bp)^{3/2}},$$

with $\lambda > -2\Gamma_{\min} = 0$. Then we may verify as above that $F(\lambda)$ is strictly decreasing with $\lim_{\lambda \rightarrow 0} F(\lambda) = +\infty$ and $\lim_{\lambda \rightarrow g^2/b^2} F(\lambda) < 0$. This gives the existence of $\lambda_c \in (0, g^2/b^2)$ such that $F(\lambda_c) = 1/g$ and consequently $\gamma(0)\sqrt{\lambda_c} > -g$.

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